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The Steady-State Solution Of Serial Channel With Feedback And Reneging Connected With Non-Serial Queuing Processes With Reneging And Balking

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Abstract - This Paper considers the steady-state behavior of serial queuing processes with feedback and reneging connected with non-serial queuing channels having reneging and balking in which (i) M serial queuing process with feedback and reneging are connected with N non-serial channel with reneging and balking. (ii) A customer may join any channel from outside and leave the system at any stage after getting service. (iii) Feedback is permitted from each channel to its previous channel in serial channels. (iv) The impatient customer may leave the service facility in serial and non-serial channel after a wait of certain time and balking has been incorporated in non-serial channels only. (v) The Input process depend upon queue size in non-serial channels and Poisson arrivals and exponential service times are followed. (vi) The queue discipline is random selection for service (vii) Waiting space is infinite. Expressions for mean queue lengths have been derived.
Key Words : Steady-State, waiting space, serial, non-serial, random selection, Poisson arrivals, exponential service, feedback, balking, reneging, marginal probabilities and mean queue length.

1 Introduction

O'Brien [6], Jackson [3] and Hunt [4] studied the problems of serial queues in the steady state with Poisson assumptions. In these studies, it is assumed that the unit must go through each service channel without leaving the system. Barrer [1] obtained the steady-state solution of a single channel queuing model having Poisson input, exponential holding time, random selection introducing reneging. Finch [2] studied simple queues with customers at random for service at a number of service stations in series with feedback. Singh [8] studied the problem of serial queues introducing the concept of reneging. Punam, Singh and Ashok [7] found the steady-state solution of serial queuing processes where feedback is not permitted.

2. Formulation of the Model

The system consists of the serial queues $Q_j\left(j=1,2,3,....,M\right)$ and non-serial channels $Q_{1i}(i=1,2,3,....N)$ with respective servers $S_j\left(j=1,2,3,....,M\right)$ and $S_{1i}\left(i=1,2,3,....N\right)$. Customers demanding different types of service arrive from outside the system in Poisson stream with parameters $\lambda_j\left(j=1,2,3,......,M\right)$ and $\lambda_{1i}\left(i=1,2,3,...N\right)$ at $Q_j\left(j=1,2,3,....,M\right)$ and $Q_{1i}(i=1,2,3,....N)$ but the sight of long queue at $Q_{1i}(i=1,2,3,....N)$ may discourage the fresh customer from joining it and may decide not to enter the service channel at $Q_{1i}(i=1,2,3,....N)$. Then the Poisson input rate at $Q_{1i}(i=1,2,3,....N)$ would be $\frac{\lambda_{1i}}{m_i+1}$ where m_i is the queue size of $Q_{1i}\left(i=1,2,3,....,N\right)$. Further, the impatient customer joining any service channel $Q_j\left(j=1,2,3,....,M\right)$ and $Q_{1i}(i=1,2,3,....N)$ may leave the queue without getting service after wait of certain time. Service time distributions for servers $S_j\left(j=1,2,3,....,M\right)$ and $S_{1i}\left(i=1,2,3,....N\right)$ are mutually independent negative exponential distribution with parameters $\mu_j\left(j=1,2,......,M\right)$ and $\mu_{1i}\left(i=1,2,3,...N\right)$ respectively. After the completion of service at S_j, the customer either leaves the system with probability p_j or joins the next channel with probability q_j or join back the previous channel with probability r_j such that $p_j+q_j+r_j=1$ $\left(j=1,2,3,...M-1\right)$ and after the completion of service at S_M the customer either leaves the system with probability p_M or join back the previous channel with probability r_M or join any queue $Q_{1i}\left(i=1,2,3,.....N\right)$ with probability $\frac{q_{Mi}}{m_i+1}\left(i=1,2,3,....N\right)$

such that $p_M+r_M+\sum_{i=1}^{N}\frac{q_{Mi}}{m_i+1}=1$

It is being mentioned here that $r_j=0$ for $j=1$ as there is no previous channel of the first channel.

The applications of such models are of common occurrence. For example, consider the administration of a particular district in a particular state at the level of district head quarter consisting of Block Development officer, Tehsildar, Sub-Divisional Magistrate, District Magistrate etc. These officers correspond to the servers of serial channels. Education Department, Health Department, Irrigation Department etc. connected with the last server of serial queue correspond to non-serial channels. The people meet the officers of the district in connection with their problems. It is also a common practice that officers call the customers (people) for hearing randomly. The senior officer may send any customer to his junior if some information regarding the customer's problem is lacking. Further District Magistrate may send the customers to different departments such as Education, Health, Irrigation etc if there problems are related to such departments. The customer after seeing long queues before any non-serial service channel may decide not to enter the queue. It generally happens that person becomes impatient after joining the queue and may leave the channel without getting service.

3. **Formulation of Equations**:

Define: $P(n_1, n_2, n_3, ...n_{M-1}, n_M, m_1, m_2, m_3, ...m_{N-1}, m_N; t)$ = the probability that at time 't' there are n_j customers (which may leave the system after service or join the next phase or join back the previous channel or renege) waiting before $S_j\,(j = 1, 2, 3, ..., M-1, M)$; m_i customers (which may balk or renege) waiting before the severs $S_{1i}\,(i = 1, 2, 3, ...N)$

We define the operators $T_i\cdot$, $T\cdot_i$, $T\cdot_{i,i+1}\cdot$, $T_{i-1}\cdot,\cdot_i$ to act upon the vectors $\tilde{n} = (n_1, n_2, n_3, ...n_M)$ or $\tilde{m} = (m_1, m_2, m_3, ...m_N)$ as fallows

$$T_i\cdot(\tilde{n}) = (n_1, n_2, n_3, ..., n_i - 1, ...n_M)$$

$$T\cdot_i(\tilde{n}) = (n_1, n_2, n_3, ..., n_i + 1, ..., n_M)$$

$$T\cdot_i,_{i+1}\cdot(\tilde{n}) = (n_1, n_2, n_3, ..., n_i + 1, n_{i+1} - 1, ..., n_M)$$

$$T_{i-1}\cdot,\cdot_i(\tilde{n}) = (n_1, n_2, n_3, ..., n_{i-1} - 1, n_i + 1...., n_M)$$

Following the procedure given by Kelly [5], we write the difference – differential equations as

$$\frac{d\,P(\tilde{n},\tilde{m};t)}{dt} = -\begin{bmatrix} \sum_{i=1}^{M}\lambda_i + \sum_{i=1}^{M}\delta(n_i)(\mu_i + C_{in_i}) \\ +\sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j+1} + \sum_{j=1}^{N}\delta(m_j)(\mu_{1j} + d_{jm_j}) \end{bmatrix} P(\tilde{n},\tilde{m};t)$$

$$+\sum_{i=1}^{M}\lambda_i P(T_i\cdot(\tilde{n}),\tilde{m};t) + \sum_{i=1}^{M}(\mu_i p_i + C_{in_i+1}) P(T\cdot_i(\tilde{n}),\tilde{m};t)$$

$$+\sum_{i=1}^{M-1}\mu_i\, q_i\, P(T\cdot_i,_{i+1}\cdot(\tilde{n}),\tilde{m};t) + \sum_{i=1}^{M}\mu_i\, r_i\, P(T_{i-1}\cdot,\cdot_i(\tilde{n}),\tilde{m};t).$$

$$+\sum_{j=1}^{N}\frac{\mu_M q_{Mj}}{m_j} P(n_1, n_2, ...n_M + 1, T_j\cdot(\tilde{m}); t) \qquad (2.1)$$

$$+\sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j} P(\tilde{n}, T_j\cdot(\tilde{m}); t) + \sum_{j=1}^{N}(\mu_{1j} + d_{jm_j+1}) P(\tilde{n}, T\cdot_j(\tilde{m}); t)$$

For $n_i \geq 0 \quad (i = 1, 2, 3, ..., M)$, $m_j \geq 0\,(j = 1, 2, 3, ...N)$;

where

$$\delta(x) = \begin{cases} 1 & if \quad x \neq 0 \\ 0 & if \quad x = 0 \end{cases}$$

and $P(\tilde{n},\tilde{m};t) = \tilde{0}$ if any of the arguments in negative.

4. Steady–State Equations

We write the following Steady–state equations of the queuing model by equating the time-derivates to zero in the equation (2.1)

$$\left[\sum_{i=1}^{M}\lambda_i + \sum_{i=1}^{M}\delta(n_i)(\mu_i + C_{in_i}) + \sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j+1} + \sum_{j=1}^{N}\delta(m_j)(\mu_{1j} + d_{jm_j})\right] P(\tilde{n},\tilde{m})$$

$$= \sum_{i=1}^{M}\lambda_i P(T_i\cdot(\tilde{n}),\tilde{m}) + \sum_{i=1}^{M}(\mu_i p_i + C_{in_i+1}) P(T\cdot_i(\tilde{n}),\tilde{m})$$

$$+\sum_{i=1}^{M-1}\mu_i\, q_i\, P(T\cdot_i,_{i+1}\cdot(\tilde{n}),\tilde{m}) + \sum_{i=1}^{M}\mu_i\, r_i\, P(T_{i-1}\cdot,\cdot_i(\tilde{n}),\tilde{m}).$$

$$+\sum_{j=1}^{N}\frac{\mu_M q_{Mj}}{m_j} P(n_1, n_2, ...n_M + 1, T_j\cdot(\tilde{m}))$$

$$(3.1)$$

$$+\sum_{j=1}^{N}\frac{\lambda_{1j}}{m_j} P(\tilde{n}, T_j\cdot(\tilde{m})) + \sum_{j=1}^{N}(\mu_{1j} + d_{jm_j+1}) P(\tilde{n}, T\cdot_j(\tilde{m}))$$

For $n_i \geq 0 \quad (i = 1, 2, 3, ..., M)$,

$m_j \geq 0\,(j = 1, 2, 3, ...N)$;

5. Steady-State Solutions

The solutions of the Steady-State equations (3.1) can be verified to be

$$P\left(\tilde{n},\tilde{m}\right)=P\left(\tilde{0},\tilde{0}\right)\left(\frac{\left(\lambda_1+\frac{\mu_2 r_2\rho_2}{\mu_2+C_{2n_2+1}}\right)^{n_1}}{\prod_{i=1}^{n_1}\left(\mu_1+C_{1i}\right)}\right).$$

$$\cdot\left(\frac{\left(\lambda_2+\frac{\mu_1 q_1\rho_1}{\mu_1+C_{1n_1+1}}+\frac{\mu_3 r_3\rho_3}{\mu_3+C_{3n_3+1}}\right)^{n_2}}{\prod_{i=1}^{n_2}\left(\mu_2+C_{2i}\right)}\right)\cdot\left(\frac{\left(\lambda_3+\frac{\mu_2 q_2\rho_2}{\mu_2+C_{2n_2+1}}+\frac{\mu_4 r_4\rho_4}{\mu_4+C_{4n_4+1}}\right)^{n_3}}{\prod_{i=1}^{n_3}\left(\mu_3+C_{3i}\right)}\right)\cdots$$

$$\cdot\left(\frac{\left(\lambda_{M-1}+\frac{\mu_{M-2} q_{M-2}\rho_{M-2}}{\mu_{M-2}+C_{M-2n_{M-2}+1}}+\frac{\mu_M r_M\rho_M}{\mu_M+C_{Mn_M+1}}\right)^{n_{M-1}}}{\prod_{i=1}^{n_{M-1}}\left(\mu_{M-1}+C_{M-1i}\right)}\right).$$

$$\cdot\left(\left(\frac{\left(\lambda_M+\frac{\mu_{M-1} q_{M-1}\rho_{M-1}}{\mu_{M-1}+C_{M-1n_{M-1}+1}}\right)^{n_M}}{\prod_{i=1}^{n_M}\left(\mu_M+C_{Mi}\right)}\right)\right).$$

$$\cdot\left(\frac{\left(\lambda_{11}+\mu_M q_{M1}\rho_M\right)^{m_1}}{m_1!\prod_{j=1}^{m_1}\left(\mu_{11}+d_{1j}\right)}\right)\left(\frac{\left(\lambda_{12}+\mu_M q_{M2}\rho_M\right)^{m_2}}{m_2!\prod_{j=1}^{m_2}\left(\mu_{12}+d_{2j}\right)}\right)\cdots$$

$$\cdot\left(\frac{\left(\lambda_{1N}+\mu_M q_{MN}\rho_M\right)^{m_N}}{m_N!\prod_{j=1}^{m_N}\left(\mu_{1N}+d_{Nj}\right)}\right)\qquad(4.1)$$

$n_i\geq 0\quad\left(i=1,2,3,\ldots,M\right)$, $m_j\geq 0\left(j=1,2,3,\ldots N\right)$

Where

$$\rho_1=\lambda_1+\frac{\mu_2 r_2\rho_2}{\mu_2+C_{2n_2+1}}$$

$$\rho_2=\lambda_2+\frac{\mu_1 q_1\rho_1}{\mu_1+C_{1n_1+1}}+\frac{\mu_3 r_3\rho_3}{\mu_3+C_{3n_3+1}}$$

$$\rho_3=\lambda_3+\frac{\mu_2 q_2\rho_2}{\mu_2+C_{2n_2+1}}+\frac{\mu_4 r_4\rho_4}{\mu_4+C_{4n_4+1}}\qquad(4.2)$$

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$$\rho_{M-1}=\lambda_{M-1}+\frac{\mu_{M-2} q_{M-2}\rho_{M-2}}{\mu_{M-2}+C_{M-2n_{M-2}+1}}+\frac{\mu_M r_M\rho_M}{\mu_M+C_{Mn_M+1}}$$

$$\rho_M=\lambda_M+\frac{\mu_{M-1} q_{M-1}\rho_{M-1}}{\mu_{M-1}+C_{M-1n_{M-1}+1}}$$

Solving these (4.2) M-equations for ρ_M with the help of determinants, we get

$$\rho_M=\frac{\left(\begin{array}{l}\lambda_M\Delta_{M-1}+\frac{q_{M-1}\mu_{M-1}}{\mu_{M-1}+C_{M-1n_{M-1}+1}}\lambda_{M-1}\Delta_{M-2}\\+\frac{q_{M-1}\mu_{M-1}}{\mu_{M-1}+C_{M-1n_{M-1}+1}}\frac{q_{M-2}\mu_{M-2}}{\mu_{M-2}+C_{M-2n_{M-2}+1}}\lambda_{M-2}\Delta_{M-3}+\ldots\ldots\\+\frac{q_{M-1}\mu_{M-1}}{\mu_{M-1}+C_{M-1n_{M-1}+1}}\frac{q_{M-2}\mu_{M-2}}{\mu_{M-2}+C_{M-2n_{M-2}+1}}\cdots\cdots\frac{q_3\mu_3}{\mu_3+C_{3n_3+1}}\lambda_3\Delta_2\\+\frac{q_{M-1}\mu_{M-1}}{\mu_{M-1}+C_{M-1n_{M-1}+1}}\frac{q_{M-2}\mu_{M-2}}{\mu_{M-2}+C_{M-2n_{M-2}+1}}\cdots\frac{q_2\mu_2}{\mu_2+C_{2n_2+1}}\lambda_2\Delta_1\\+\frac{q_{M-1}\mu_{M-1}}{\mu_{M-1}+C_{M-1n_{M-1}+1}}\frac{q_{M-2}\mu_{M-2}}{\mu_{M-2}+C_{M-2n_{M-2}+1}}\cdots\frac{q_2\mu_2}{\mu_2+C_{2n_2+1}}\frac{q_1\mu_1}{\mu_1+C_{1n_1+1}}\lambda_1\end{array}\right)}{\Delta_M}$$

(4.3)

where $\Delta_M=\Delta_{M-1}-\frac{q_{M-1}\mu_{M-1}}{\mu_{M-1}+C_{M-1n_{M-1}+1}}\frac{r_M\mu_M}{\mu_M+C_{Mn_M+1}}\Delta_{M-2}$

$$\Delta_{M-1}=\Delta_{M-2}-\frac{q_{M-2}\mu_{M-2}}{\mu_{M-2}+C_{M-2n_{M-2}+1}}\frac{r_{M-1}\mu_{M-1}}{\mu_{M-1}+C_{M-1n_{M-1}+1}}\Delta_{M-3}\qquad(4.4)$$

Continuing in this way

$$\Delta_3=\Delta_2-\frac{q_2\mu_2}{\mu_2+C_{2n_2+1}}\frac{r_3\mu_3}{\mu_3+C_{3n_3+1}}$$

Where

$$\Delta_M=\begin{vmatrix}1&\frac{-r_2\mu_2}{\mu_2+C_{2n_2+1}}&0&0&--&0&0&0\\\frac{-q_1\mu_1}{\mu_1+C_{1n_1+1}}&1&\frac{-r_3\mu_3}{\mu_3+C_{3n_3+1}}&0&--&0&0&0\\0&\frac{-q_2\mu_2}{\mu_2+C_{2n_2+1}}&1&\frac{-r_4\mu_4}{\mu_4+C_{4n_4+1}}&--&0&0&0\\-&-&-&-&--&-&-&-\\-&-&-&-&--&-&-&-\\-&-&-&-&--&-&0&0\\0&0&0&0&--&\frac{-q_{M-2}\mu_{M-2}}{\mu_{M-2}+C_{M-2n_{M-2}+1}}&1&\frac{-r_M\mu_M}{\mu_M+C_{Mn_M+1}}\\0&0&0&0&--&0&\frac{-q_{M-1}\mu_{M-1}}{\mu_{M-1}+C_{M-1n_{M-1}+1}}&1\end{vmatrix}$$

$$\Delta_2=\begin{vmatrix}1&\frac{-r_2\mu_2}{\mu_2+C_{2n_2+1}}\\\frac{-q_1\mu_1}{\mu_1+C_{1n_1+1}}&1\end{vmatrix}=1-\frac{r_2\mu_2}{\mu_2+C_{2n_2+1}}\frac{q_1\mu_1}{\mu_1+C_{1n_1+1}}$$

$\Delta_1=\left|1\right|=1$

Since ρ_M is obtained, we can get ρ_{M-1} by putting the value of ρ_M in the last equation of (4.2), ρ_{M-2} by putting the values of ρ_{M-1} and ρ_M in the last but one equation of (4.2) continuing in this way, we shall obtain $\rho_{M-3},\rho_{M-4},---,\rho_3,\rho_2$, and ρ_1,.

Thus, we write (4.1) as under

$$P(\tilde{n},\tilde{m}) = P(\tilde{0},\tilde{0})\left(\frac{(\rho_1)^{n_1}}{\prod_{i=1}^{n_1}\mu_1 + C_{1i}}\right)\left(\frac{(\rho_2)^{n_2}}{\prod_{i=1}^{n_2}\mu_2 + C_{2i}}\right).$$

$$.\left(\frac{(\rho_3)^{n_3}}{\prod_{i=1}^{n_3}\mu_3 + C_{3i}}\right)---\left(\frac{(\rho_{M-1})^{n_{M-1}}}{\prod_{i=1}^{n_{M-1}}\mu_{M-1} + C_{M-1i}}\right)\left(\frac{(\rho_M)^{n_M}}{\prod_{i=1}^{n_M}\mu_M + C_{Mi}}\right)$$

$$\left(\frac{(\lambda_{11} + \mu_M q_{M1}\rho_M)^{m_1}}{m_1!\prod_{j=1}^{m_1}(\mu_{11} + d_{1j})}\right).\left(\frac{(\lambda_{12} + \mu_M q_{M2}\rho_M)^{m_2}}{m_2!\prod_{j=1}^{m_2}(\mu_{12} + d_{2j})}\right)....\left(\frac{(\lambda_{1N} + \mu_M q_{MN}\rho_M)^{m_N}}{m_N!\prod_{j=1}^{m_N}(\mu_{1N} + d_{Nj})}\right).$$

$$for\ n_i \geq 0,(i = 1,2,—,M)\ ,m_j\ (j = 1,2,3,....N) \qquad (4.5)$$

We obtain $P(\tilde{0},\tilde{0})$ from the normalizing conditions.

$$\sum_{\tilde{n}=\tilde{0},\tilde{m}=\tilde{0}}^{\infty} P(\tilde{n},\tilde{m}) = 1 \qquad (4.6)$$

and with the restriction that traffic intensity of each service channel of the system is less than unity,

C_{in_i} and d_{jm_j} are the reneging rates at which customers renege after a wait of time T_{0i} whenever there are n_i and m_j customers in the service channel Q_i and Q_{1j}.

$$C_{in_i} = \frac{\mu_{1i}e^{-\frac{\mu_{1i}T_{0i}}{n_i}}}{1-e^{-\frac{\mu_{1i}T_{0i}}{n_i}}} \quad (i = 1,2,3,....M)$$

and $d_{jm_j} = \dfrac{\mu_{1j}e^{-\frac{\mu_{1j}T_{0i}}{m_j}}}{1-e^{-\frac{\mu_{1j}T_{0i}}{m_j}}} \quad (j = 1,2,3,....N)$

Here it is mentioned that the customers leave the system at constant rate as long as there is a line, provided that the customers are served in the ordered in which they arrive. Putting

$C_{in_i} = C_i \qquad (i = 1,2,3,...M)$

and $d_{jm_j} = d_j \qquad (j = 1,2,3,.....N)$ in the steady-state solution (4.1)we get

$$P(\tilde{n},\tilde{m}) = P(\tilde{0},\tilde{0})\left(\frac{\rho_1}{\mu_1 + C_1}\right)^{n_1}\left(\frac{\rho_2}{\mu_2 + C_2}\right)^{n_2}\left(\frac{\rho_3}{\mu_3 + C_3}\right)^{n_3}..$$

$$..\left(\frac{\rho_{M-1}}{\mu_{M-1} + C_{M-1}}\right)^{n_{M-1}}\left(\frac{\rho_M}{\mu_M + C_M}\right)^{n_M}.$$

$$.\left(\frac{1}{m_1!}\left(\frac{\lambda_{11} + \mu_M q_{M1}\rho_M}{\mu_{11} + d_1}\right)^{m_1}\right)\left(\frac{1}{m_2!}\left(\frac{\lambda_{12} + \mu_M q_{M2}\rho_M}{\mu_{12} + d_2}\right)^{m_2}\right)$$

$$\left(\frac{1}{m_3!}\left(\frac{\lambda_{13} + \mu_M q_{M3}\rho_M}{\mu_{13} + d_3}\right)^{m_3}\right).....\left(\frac{1}{m_N!}\left(\frac{\lambda_{1N} + \mu_M q_{MN}\rho_M}{\mu_{1N} + d_N}\right)^{m_N}\right) \qquad (4.7)$$

We obtain $P(\tilde{0},\tilde{0})$ from (4.6) and (4.7) as

$$\left(P(\tilde{0},\tilde{0})\right)^{-1} = \prod_{i=1}^{M}\left(\frac{1}{1-\frac{\rho_i}{\mu_i + C_i}}\right)\prod_{j=1}^{N}e^{\rho_{1j}}$$

Where $\rho_{1j} = \dfrac{\lambda_{1j} + \mu_M q_{Mj}\rho_M}{\mu_{1j} + d_j}$, $j = 1,2,3,...N$

Thus $P(\tilde{n},\tilde{m})$ is completely determined.

6. Steady-State Marginal Probabilities

Let $P(n_1)$ be the steady-state marginal probability that there are n_1 units in the queue before the first server. This is determined as

$$P(n_1) = \sum_{n_2,n_3,....n_{M=0}}^{\infty}\sum_{\tilde{m}=\tilde{0}}^{\infty}P(\tilde{n},\tilde{m})$$

$$= \sum_{n_2,n_3,....n_{M=0}}^{\infty}\sum_{\tilde{m}=\tilde{0}}^{\infty}P(\tilde{0},\tilde{0})\left(\frac{\rho_1}{\mu_1 + C_1}\right)^{n_1}\left(\frac{\rho_2}{\mu_2 + C_2}\right)^{n_2}\left(\frac{\rho_3}{\mu_3 + C_3}\right)^{n_3}....$$

$$\left(\frac{\rho_{M-1}}{\mu_{M-1} + C_{M-1}}\right)^{n_{M-1}}\left(\frac{\rho_M}{\mu_M + C_M}\right)^{n_M}.$$

$$.\left(\frac{1}{m_1!}\left(\frac{\lambda_{11} + \mu_M q_{M1}\rho_M}{\mu_{11} + d_1}\right)^{m_1}\right)\left(\frac{1}{m_2!}\left(\frac{\lambda_{12} + \mu_M q_{M2}\rho_M}{\mu_{12} + d_2}\right)^{m_2}\right)$$

$$\left(\frac{1}{m_3!}\left(\frac{\lambda_{13} + \mu_M q_{M3}\rho_M}{\mu_{13} + d_3}\right)^{m_3}\right).....\left(\frac{1}{m_N!}\left(\frac{\lambda_{1N} + \mu_M q_{MN}\rho_M}{\mu_{1N} + d_N}\right)^{m_N}\right)$$

Thus $P(n_1) = \left(\dfrac{\rho_1}{\mu_1 + C_1}\right)^{n_1}\left(1-\dfrac{\rho_1}{\mu_1 + C_1}\right) \qquad n_1 > 0$

Similarly

$$P(n_2) = \left(\frac{\rho_2}{\mu_2 + C_2}\right)^{n_2}\left(1-\frac{\rho_2}{\mu_2 + C_2}\right) \qquad n_2 > 0$$

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$$P(n_M) = \left(\frac{\rho_M}{\mu_M + C_M}\right)^{n_M}\left(1 - \frac{\rho_M}{\mu_M + C_M}\right) \qquad n_M > 0$$

Further, let $P(m_1), P(m_2), P(m_3), \ldots, P(m_N)$ be the steady-state marginal probabilities that there are $m_1, m_2, m_3, \ldots, m_N$ customers waiting before server $S_{1i} (i = 1, 2, 3, \ldots, N)$ respectively.

$$P(m_1) = \sum_{\tilde{n}=\tilde{0}}^{\infty} \sum_{m_2, m_3, \ldots m_N = 0}^{\infty} P(\tilde{n}, \tilde{m})$$

$$= \sum_{\tilde{n}=\tilde{0}}^{\infty} \sum_{m_2, m_3, \ldots m_N = 0}^{\infty} P(\tilde{0}, \tilde{0})\left(\frac{\rho_1}{\mu_1 + C_1}\right)^{n_1}\left(\frac{\rho_2}{\mu_2 + C_2}\right)^{n_2}\left(\frac{\rho_3}{\mu_3 + C_3}\right)^{n_3} \cdots$$

$$\cdot\left(\frac{\rho_{M-1}}{\mu_{M-1} + C_{M-1}}\right)^{n_{M-1}}\left(\frac{\rho_M}{\mu_M + C_M}\right)^{n_M}\left(\frac{1}{m_1!}(\rho_{11})^{m_1}\right).$$

$$\cdot\left(\frac{1}{m_2!}(\rho_{12})^{m_2}\right)\cdots\left(\frac{1}{m_N!}(\rho_{1N})^{m_N}\right)$$

$$P(m_1) = \frac{1}{m_1!}\frac{(\rho_{11})^{m_1}}{e^{11}}; \qquad m_1 > 0$$

Similarly

$$P(m_2) = \frac{1}{m_2!}\frac{(\rho_{12})^{m_2}}{e^{\rho_{12}}}; \qquad m_2 > 0$$

....

....

$$P(m_N) = \frac{1}{m_N!}\frac{(\rho_{1N})^{m_N}}{e^{\rho_{1N}}}; \qquad m_N > 0$$

7. Mean Queue Length

Mean queue length before the server S_1 is determined by

$$L_1 = \sum_{n_1=0}^{\infty} n_1 P(n_1) = \sum_{n_1=0}^{\infty} n_1 \left(\frac{\rho_1}{\mu_1 + C_1}\right)^{n_1}\left(1 - \frac{\rho_1}{\mu_1 + C_1}\right)$$

$$= \left(\frac{\rho_1}{\mu_1 + C_1 - \rho_1}\right)$$

Similarly

$$L_2 = \left(\frac{\rho_2}{\mu_2 + C_2 - \rho_2}\right)$$

$$L_M = \frac{\rho_M}{\mu_M + C_M - \rho_M}$$

Mean queue length before the server S_{11} is determined as

$$L_{11} = \rho_{11}$$

Similarly

$$L_{1j} = \rho_{1j}; \qquad j = 2, 3, \ldots, N$$

Hence mean queue length of the system is

$$L = \sum_{k=1}^{M} L_k + \sum_{j=1}^{N} L_{1j}$$

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